Teleparallel Equivalent of Kaluza-Klein

V. C. de Andrade, L. C. T. Guillen and J. G. Pereira
Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona 145
01405-900 São Paulo SP
Brazil

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Relying upon the equivalence between a gauge theory for the translation group and general relativity, a teleparallel version of the original Kaluza-Klein theory is developed. In this model, only the internal space (fiber) turns out to be five-dimensional, spacetime being kept always four-dimensional. A five-dimensional translational gauge theory is obtained which unifies, in the sense of Kaluza-Klein, gravitational and electromagnetic interactions.

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In ordinary Kaluza-Klein theories [1], the geometrical approach of general relativity is adopted as the paradigm for the description of all other interactions of nature. In the original Kaluza-Klein theory, for example, gravitational and electromagnetic fields are described by a Hilbert-Einstein Lagrangian in a five-dimensional spacetime. On the other hand, it has already been shown that, at least macroscopically, general relativity is equivalent to a gauge theory for the translation group [2]. In this theory, known as the *teleparallel equivalent of general relativity*, the fundamental field is the Cartan connection, a connection presenting torsion, but no curvature. This equivalence gives rise to new perspectives for the study of unified theories. In fact, instead of using the geometrical description of general relativity, we can adopt the gauge description as the basic paradigm, and in this way construct what we call the *teleparallel equivalent of Kaluza-Klein*. According to this approach, both gravitational and electromagnetic fields turn out to be described by a gauge-type Lagrangian. Such a construction will be the main purpose of this letter.

Let us start by studying the teleparallel equivalent of general relativity. We denote by $x^\mu (\mu, \nu, \rho, ... = 0, 1, 2, 3)$ the coordinates of spacetime, and by $x^a (a, b, c, ... = 0, 1, 2, 3)$ the coordinates of the tangent space, assumed to be a Minkowski space with the metric

$$\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1). \tag{1}$$

The holonomous derivatives in these spaces are related by

$$\partial_\mu = (\partial_\mu x^a)\partial_a \quad \text{and} \quad \partial_a = (\partial_a x^\mu)\partial_\mu, \tag{2}$$

where $\partial_\mu x^a$ is a trivial holonomous tetrad, with $\partial_a x^\mu$ its inverse. A gauge transformation is defined as a local translation of the tangent-space (fiber) coordinates,

$$\delta x^a = \delta\alpha^b P_b x^a, \tag{3}$$

with $P_a = \partial/\partial x^a$ the translation generators, and $\delta\alpha^a$ the corresponding infinitesimal parameters.

Consider now a general source field $\Psi(x^\mu)$, whose gauge transformation is

$$\delta\Psi = \delta\alpha^a P_a \Psi. \tag{4}$$

Notice that P_a is able to act on Ψ because of the identities (2). Denoting the gauge potentials by $A^a{}_\mu$, its covariant derivative can be written as [3]

$$D_\mu \Psi = h^a{}_\mu \, \partial_a \Psi, \tag{5}$$

where

$$h^a{}_\mu = \partial_\mu x^a + c^{-2} A^a{}_\mu \tag{6}$$

is a nontrivial tetrad field, with the speed of light c introduced for dimensional reasons. From the covariance of D_μ, we can obtain the transformation of the gauge potentials:

$$A^{a'}{}_\mu = A^a{}_\mu - c^2 \partial_\mu \delta\alpha^a. \tag{7}$$

As usual in gauge theories, the field strength is given by

$$[D_\mu, D_\nu]\Psi = c^{-2} F^a{}_{\mu\nu} P_a \Psi, \tag{8}$$

with

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu. \tag{9}$$

We notice that the tangent space indices are raised and lowered with the metric η_{ab} while the spacetime indices are raised and lowered with the Riemannian metric

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu \, h^b{}_\nu. \tag{10}$$

The presence of a nontrivial tetrad field induces on spacetime a teleparallel structure which is directly related to the presence of the gravitational field. In fact, given a nontrivial tetrad $h^a{}_\mu$, it is possible to define a Cartan connection

$$\Gamma^{\rho}{}_{\mu\nu} = h_a{}^{\rho} \partial_\nu h^a{}_\mu, \tag{11}$$

which is a connection presenting torsion, but no curvature [4]. It is easy to see that the Cartan covariant derivative of the tetrad field vanishes identically:

$$\nabla_\nu h^a{}_\mu \equiv \partial_\nu h^a{}_\mu - \Gamma^{\theta}{}_{\mu\nu}\, h^a{}_\theta = 0. \tag{12}$$

This is the absolut parallelism condition. It is also easy to see that the gravitational field strength $F^a{}_{\mu\nu}$ can be rewritten in the form $F^a{}_{\mu\nu} = c^2 h^a{}_\rho T^\rho{}_{\mu\nu}$, where

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} \tag{13}$$

is the torsion of the Cartan connection.

The gauge gravitational field Lagrangian is [3]

$$\mathcal{L}_G = \frac{h}{16\pi G} \left(\frac{1}{4} F^a{}_{\mu\nu} F^b{}_{\theta\rho} g^{\mu\theta} N_{ab}{}^{\nu\rho} \right), \tag{14}$$

where $h = \det(h^a{}_\mu)$. In principle, one should have $N_{ab}{}^{\nu\rho} = \eta_{ab}\; h_c{}^\nu\; h^{c\rho}$. However, due to the presence of the tetrads, spacetime and tangent-space indices can be changed into each other. This means that $N_{ab}{}^{\nu\rho}$ must include all cyclic permutation of a, b and c:

$$N_{ab}{}^{\nu\rho} = \eta_{ab}\; h_c{}^\nu\; h^{c\rho} + 2\, h_a{}^\rho\; h_b{}^\nu - 4\, h_a{}^\nu\; h_b{}^\rho. \tag{15}$$

Substituting (15) into (14), we get

$$\mathcal{L}_G \equiv h L_G = \frac{h c^4}{16\pi G}\, S^{\rho\mu\nu}\, T_{\rho\mu\nu}, \tag{16}$$

where

$$S^{\rho\mu\nu} = \frac{1}{2} \left[K^{\mu\nu\rho} - g^{\rho\nu}\, T^{\theta\mu}{}_\theta + g^{\rho\mu}\, T^{\theta\nu}{}_\theta \right],$$

with $K^{\mu\nu\rho}$ the contorsion tensor. By considering now the functional variation of $\mathcal{L}_G$ in relation to $A^a{}_\mu$, we obtain the (teleparallel) gravitational field equation

$$\partial_\sigma (h S_\lambda{}^{\sigma\tau}) - \frac{4\pi G}{c^4}\, (h t_\lambda{}^\tau) = 0, \tag{17}$$

with

$$t_\lambda{}^\tau = \frac{c^4}{4\pi G}\, \Gamma^{\mu}{}_{\nu\lambda} S_\mu{}^{\nu\tau} + \delta_\lambda{}^\tau L_G \tag{18}$$

the canonical energy-momentum (pseudo) tensor of the gravitational field. It is important to remark that Eq.(17) is symmetric in $(\lambda\ \tau)$.

As already remarked, the curvature of the Cartan connection vanishes identically:

$$R^\theta{}_{\rho\mu\nu} = \partial_\mu \Gamma^\theta{}_{\rho\nu} + \Gamma^\theta{}_{\sigma\mu}\, \Gamma^\sigma{}_{\rho\nu} - (\mu \leftrightarrow \nu) \equiv 0. \tag{19}$$

Substituting the relation

$$\Gamma^\theta{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\theta{}_{\mu\nu} + K^\theta{}_{\mu\nu}, \tag{20}$$

with $\overset{\circ}{\Gamma}{}^\theta{}_{\mu\nu}$ the Levi-Civita connection of the metric $g_{\mu\nu}$, we get

$$R^\theta{}_{\rho\mu\nu} = \overset{\circ}{R}{}^\theta{}_{\rho\mu\nu} + \Big[\nabla_\mu K^\theta{}_{\rho\nu} + K^\sigma{}_{\rho\mu} K^\theta{}_{\sigma\nu} + K^\sigma{}_{\nu\mu} K^\theta{}_{\rho\sigma} - (\mu \leftrightarrow \nu) \Big] \equiv 0 \tag{21}$$

where ∇_μ is the Cartan covariant derivative defined in (12), and $\overset{\circ}{R}{}^\theta{}_{\rho\mu\nu}$ is the curvature of the Levi-Civita connection. If we compute from this equation the scalar curvature $\overset{\circ}{R}$, and substitute in the Hilbert-Einstein Lagrangian of general relativity

$$\mathcal{L} = -\frac{c^4}{16\pi G}\, \sqrt{-g}\, \overset{\circ}{R}, \tag{22}$$

where $g = \det(g_{\mu\nu})$, up to a total divergence, we get exactly the Lagrangian (16) of a gauge theory for the translation group. Accordingly, the field equation (17) results equivalent to Einstein's equations [5].

In the framework of the teleparallel description of gravitation, the action describing a particle of mass m and charge e under the influence of both an electromagnetic and a gravitational field is [3,6]

$$S = \int_a^b \left[-m\, c\, ds - \frac{m}{c}\, A^a{}_\mu\, u_a\, dx^\mu - \frac{e}{c}\, A_\mu\; dx^\mu \right], \tag{23}$$

where $A^a{}_\mu$ is the gravitational and A_μ is the electromagnetic gauge potentials. The corresponding equation of motion is

$$c^2 h^a{}_\rho \frac{du_a}{ds} = F^a{}_{\rho\mu} u_a u^\mu + \frac{e}{m}\, F_{\rho\mu} u^\mu. \tag{24}$$

Now, inspired in the similarity between the gravitational and the electromagnetic interactions in this approach, we choose the $U(1)$ gauge index of the electromagnetic theory to be "5", which allows us to define a unified gauge potential according to

$$\mathcal{A}^A{}_\mu = \left(A^a{}_\mu, A^5{}_\mu \right),$$

where $A^5{}_\mu = \kappa^{-1}(e/m) A_\mu$, with κ a dimensionless parameter to be determined later. Accordingly, we can define a unified field strength,

$$\mathcal{F}^A{}_{\mu\nu} = \left(F^a{}_{\mu\nu}, F^5{}_{\mu\nu} \right),$$

with $F^5{}_{\mu\nu} = \kappa^{-1}(e/m) F_{\mu\nu}$. In terms of the potential $\mathcal{A}^A{}_\mu$, therefore, the field strength is

$$\mathcal{F}^A{}_{\mu\nu} = \partial_\mu \mathcal{A}^A{}_\nu - \partial_\nu \mathcal{A}^A{}_\mu. \tag{25}$$

Implicit in the above definitions is the introduction of an internal five-dimensional space M^5, given by the produt between the Minkowski space M^4 and the circle S^1: $M^5 = M^4 \otimes S^1$. A point in this space is determined by the coordinates $x^A = (x^a, x^5)$, where x^a are the coordinates of M^4, and $x^5 = \kappa^{-1}(e/m)x$ the coordinate of S^1. We can also define a five-velocity $u^A = (u^a, u^5)$ by

$$u^A = \frac{dx^A}{d\sigma}, \tag{26}$$

where $d\sigma^2 = \eta_{ab}dx^a dx^b$, such that u^a is the usual four-velocity, and u^5 is a strictly internal component whose value will be determined by the unification process.

With the above definitions, and denoting by η_{55} the fifth component of the internal-space metric, if the conditions $u^5 = -\kappa$ and $\eta_{55} = -1$ are satisfied, the action (23) can be rewriten in the form

$$S = \int_a^b \left[-m\ c\ ds - \frac{m}{c} \mathcal{A}^A{}_\mu\, u^B\, \eta_{AB}\, dx^\mu \right]. \tag{27}$$

Accordingly, the equation of motion (24) becomes

$$c^2 h^a{}_\rho \frac{du_a}{ds} = \mathcal{F}^A{}_{\rho\mu}\ u^B\ u^\mu\, \eta_{AB}. \tag{28}$$

The trajectory of a charged particle submitted to both an electromagnetic and a gravitational field, therefore, is described by a Lorentz-type force equation. Furthermore, differently from curvature, we see that torsion acts on particles in the same way the electromagnetic field acts on charges, that is, as a force.

It is important to notice that, alternatively, we could have chosen $u^5 = \kappa$ and $\eta_{55} = 1$, which would lead to the same action integral, and consequently to the same equation of motion. As we will see, this choice corresponds to another metric convention for the internal space. In principle, both conventions are possible. However, the unification process will introduce a constraint according to which the choice of η_{55} will depend on the metric convention adopted for the tangent Minkowski space.

In a unified teleparallel Kaluza-Klein model, a general gauge transformation is represented by a translation of the five-dimensional internal space coordinates x^A,

$$\delta x^A = \delta\alpha^B\, P_B\, x^A, \tag{29}$$

where $P_B = \partial/\partial x^B$ are the group generators, and

$$\delta\alpha^B(x^\mu) \equiv \delta\alpha^B = \left(\delta\alpha^a, \delta\alpha^5\right)$$

are the transformation parameters. Analogously to the gauge potentials, we take $\delta\alpha^5 = \kappa^{-1}(e/m)\delta\alpha$. Now, in the same way as in ordinary Kaluza-Klein models, we assume the gauge potentials $\mathcal{A}^A{}_\mu$, and consequently the tetrad $h^a{}_\mu$ and also the metric tensor $g_{\mu\nu}$, not to depend on the coordinate x^5. On the other hand, matter fields do depend on the coordinate x^5:

$$\Psi = \Psi(x^\mu, x^5)\ .$$

Let us then consider a generalized gauge translation, under which a matter field behaves as

$$\delta\Psi = \delta\alpha^A\, P_A\, \Psi. \tag{30}$$

Its covariant derivative, consequently, is

$$D_\mu \Psi = \partial_\mu \Psi + c^{-2} \mathcal{A}^A{}_\mu P_A\, \Psi, \tag{31}$$

with the gauge transformation of $\mathcal{A}^A{}_\mu$ given by

$$\delta\mathcal{A}^B{}_\rho = -c^2 \partial_\rho \delta\alpha^B. \tag{32}$$

Separating the gravitational and electromagnetic components, we get

$$D_\mu \Psi = \partial_\mu \Psi + c^{-2} A^a{}_\mu P_a \Psi + \kappa^{-1} \frac{e}{mc^2} A_\mu\, P_5\, \Psi\ . \tag{33}$$

Now, as x^5 is the coordinate of the internal manifold S^1, we assume

$$\Psi(x^\mu, x^5) = \exp\left[i\,\kappa\, \frac{2\pi}{\lambda_C}\, x^5 \right]\ \psi(x^\mu)\ , \tag{34}$$

with $\lambda_C = (h/mc)$ the Compton wave-length of the particle under consideration. Consequently, a translation in the coordinate x^5 turns out to be a $U(1)$ gauge transformation, and a translation in the coordinates x^a turns out to be a gauge transformation related to the translation group T^4. For a simultaneous translation in the five coordinates x^A, we see from (30) that

$$\delta\Psi = \delta\alpha^a\, \partial_a \Psi + \delta\alpha \left(\frac{iec}{\hbar} \right) \Psi. \tag{35}$$

The corresponding minimal couplings are given by the covariant derivative

$$D_\mu \Psi = h^a{}_\mu\, \partial_a \Psi + \frac{ie}{\hbar c}\, A_\mu\, \Psi. \tag{36}$$

By defining $A_a = h_a{}^\mu\, A_\mu$, we can rewrite (36) in the form

$$D_\mu \Psi = h^a{}_\mu\, D_a \Psi, \tag{37}$$

with $D_a \Psi$ the electromagnetic covariant derivative in Minkowski space. As usual, the commutator of covariant derivatives yields the field strength:

$$[D_\mu, D_\nu]\Psi = c^{-2} F^a{}_{\mu\nu} P_a \Psi + \frac{ie}{\hbar c} F_{\mu\nu} \Psi. \tag{38}$$

The unified Lagrangian can be written in the form

$$\mathcal{L} = \frac{h}{16\pi G} \left(\frac{1}{4} \mathcal{F}^A{}_{\mu\nu} \mathcal{F}^B{}_{\theta\rho} g^{\mu\theta} N_{AB}{}^{\nu\rho} \right). \tag{39}$$

As we have already discussed, $N_{ab}{}^{\nu\rho}$ is given by (15). On the other hand, as no tetrad exists in the electromagnetic sector,

$$N_{55}{}^{\nu\rho} = \eta_{55} h_c{}^\nu\ h^{c\rho}. \tag{40}$$

Thus, the Lagrangian (39) becomes

$$\mathcal{L} = \frac{hc^4}{16\pi G}\, S^{\rho\mu\nu}\, T_{\rho\mu\nu} + \eta_{55}\, \frac{\kappa^{-2} e^2}{16\pi G m^2} \left[\frac{h}{4} F_{\mu\nu} F^{\mu\nu}\right]. \quad (41)$$

In order the get Maxwell's Lagrangian, two conditions must be satisfied. First, it is necessary that

$$\kappa^2 = \frac{e^2}{16\pi G m^2}, \quad (42)$$

from where we see that κ^2 turns out to be proportional to the ratio between electric and gravitational forces [7]. Second, in order to have a positive-definite energy for the electromagnetic field, and to get the appropriate relative sign between the gravitational and electromagnetic Lagrangians, it is necessary that $\eta_{55} = -1$. With these conditions, the Lagrangian (41) becomes

$$\mathcal{L} \equiv h\,(L_G + L_{EM}) = \frac{hc^4}{16\pi G}\, S^{\rho\mu\nu}\, T_{\rho\mu\nu} - \frac{h}{4} F_{\mu\nu} F^{\mu\nu}.$$

That the Maxwell Lagrangian in four dimensions shows up from the Hilbert-Einstein Lagrangian in five dimensions, is usually considered as a miracle of the Kaluza-Klein theory [8]. That the Hilbert-Einstein Lagrangian of general relativity shows up from a Maxwell-type Lagrangian for a five-dimensional translation gauge theory, can be considered as the other face of the same miracle.

Now comes an important point. Since we have chosen (1) as the metric of the Minkowski space, the resulting metric of the five-dimensional internal space will be

$$\eta_{AB} = \mathrm{diag}(+1, -1, -1, -1, -1). \quad (43)$$

This means that the fifth dimension must necessarily be space-like, and the metric with signature $(3, 2)$ is excluded. On the other hand, if we had chosen $\tilde{\eta}_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ for the Minkowski space, it is easy to verify that we should have $\eta_{55} = 1$. The resulting metric of the five-dimensional internal space would be

$$\tilde{\eta}_{AB} = \mathrm{diag}(-1, +1, +1, +1, +1), \quad (44)$$

and the same conclusion would be obtained: the fifth dimension must necessarily be space-like, and the metric with signature $(3, 2)$ is excluded. The unification of the gravitational and electromagnetic Lagrangians, therefore, imposes a constraint on the metric conventions for Minkowski and for the electromagnetic internal manifold S^1. In fact, the choice between $\eta_{55} = +1$ and $\eta_{55} = -1$ depends on the metric convention adopted for the Minkowski space. As a consequence, the metric of the five-dimensional internal space turns out to be restricted to either (43) or (44). Metrics with signature $(3, 2)$, which would imply a time-like fifth dimension, are excluded.

The functional variation of $\mathcal{L}$ in relation to $A^a{}_\mu$ leads to the field equation

$$\partial_\sigma (h S_\lambda{}^{\sigma\tau}) - \frac{4\pi G}{c^4} (h t_\lambda{}^\tau) = \frac{4\pi G}{c^4}\, (h \theta_\lambda{}^\tau), \quad (45)$$

with

$$\theta_\lambda{}^\tau \equiv h^a{}_\lambda \left[-\frac{1}{h} \frac{\delta \mathcal{L}_{EM}}{\delta h^a{}_\tau}\right] = F_{\lambda\nu} F^{\tau\nu} - \delta_\lambda{}^\tau L_{EM} \quad (46)$$

the energy-momentum tensor of the electromagnetic field. On the other hand, the functional variation of $\mathcal{L}$ in relation to A_μ yields the teleparallel version of Maxwell's equation [9].

Summing up: by replacing the general relativity paradigm by a gauge paradigm, a five-dimensional Maxwell-type translational gauge theory on a four-dimensional spacetime has been constructed. In this theory, gravitation is attributed to torsion, and the electromagnetic field strength appears as the fifth gauge-component of the torsion tensor. Furthermore, due to the fact that torsion, like the electromagnetic field, plays the role of a force, the unification in this approach seems to be much more natural than in ordinary Kaluza-Klein theories. In addition, no scalar field is generated by the unification process. It should be mentioned, finally, that several different Kaluza-Klein models in spacetimes with torsion have already been considered [10]. However, all of them are constructed on a 5-dimensional spacetime, and are consequently completely different from the model presented here.

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